TOPOLOGICITY OF BORNOLOGICAL CONVERGENCES

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Abstract

Through all this work, we will mainly deal with quasi-uniform spaces. Recall that a *quasi-uniformity* on a nonempty set X [9, 11] is a filter $\mathcal{U}$ of reflexive relations such that if $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = \{(x, z) \in X \times X :$ there exists $y \in X$ with $(x, y), (y, z) \in V\}$. Given $U \in \mathcal{U}$, we also write $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$.

Recall that a hypertopology is a topology defined over a certain family of sets. Our basic references for hypertopologies are [1, 18].

Vietoris [22, 17] defined the so-called *finite topology* on a topological space (X, τ) which is usually known as the *Vietoris topology.* On the family $\mathcal{P}_0(X)$ of all nonempty subsets of X, this topology τ_V has as a base all sets of the form

$$G^+ \cap V_1^- \cap \ldots \cap V_n^-$$

where $G, V_1, \ldots, V_n$ are open sets and

$$\begin{aligned} G^+ =&\{A \in \mathcal{P}_0(X) : A \subseteq G\} \\ V_i^- =&\{A \in \mathcal{P}_0(X) : A \cap V_i \neq \varnothing\} \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. Fell [8] considered a slight although very important modification of the above topology. In this way, the *Fell topology* τ_F has as a base all sets of the form $G^+ \cap V_1^- \cap \ldots \cap V_n^-$ where $G, V_1, \ldots, V_n$ are open sets and G^c is compact.

Notice that the only difference between τ_V and τ_F relies on the family to which the complement of G belongs: the closed sets in the case of the Vietoris topology and the closed and compact sets in the case of the Fell topology.

These two topologies follows a general pattern which was studied by Poppe [20]. Let Δ be a cobase, i. e. a family of closed sets containing the empty set, the singletons and closed under finite unions. Then the Δ-*hit-and-miss topology* has as a base all sets of the form $G^+ \cap V_1^- \cap \ldots \cap V_n^-$ where $V_1, \ldots, V_n$ are open sets and $G^c \in \Delta$.

In the literature about hypertopologies, the most well-known topology is the so-called *topology of the Hausdorff distance.* Although this topology was first defined on a metric space, it was subsequently extended to a uniform space [7] and to a quasi-uniform space [6, 13]. In this way, given a quasi-uniform space $(X, \mathcal{U})$, for each $U \in \mathcal{U}$ define

$$\begin{aligned} U_H^+ =&\{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\} \\ U_H^- =&\{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}. \end{aligned}$$

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Then $\{U_H^+ : U \in \mathcal{U}\}$ is a base for the *upper Hausdorff quasi-uniformity* $\mathcal{U}_H^+$ on $\mathcal{P}_0(X)$ and $\{U_H^- : U \in \mathcal{U}\}$ is a base for the *lower Hausdorff quasi-uniformity* $\mathcal{U}_H^-$ on $\mathcal{P}_0(X)$. The quasi-uniformity $\mathcal{U}_H = \mathcal{U}_H^+ \vee \mathcal{U}_H^-$ is the so-called *Hausdorff (or Bourbaki) quasi-uniformity* of $(X,\mathcal{U})$ on $\mathcal{P}_0(X)$.

We observe that a net $\{A_\lambda\}_{\lambda\in\Lambda}$ is convergent to A in the topology $\tau(\mathcal{U}_H)$ generated by the Hausdorff quasi-uniformity if and only if for all $U \in \mathcal{U}$

$$A_\lambda \subseteq U(A) \text{ and } A \subseteq U^{-1}(A_\lambda) \text{ residually.}$$

Nevertheless, in general, the topology generated by the Hausdorff quasi-uniformity is considered to be too strong. For example, let us consider $\mathbb{R}^2$ endowed with the usual uniformity. Then the graphs of the lines of slope $1/n$ passing through the origin form a sequence which is not convergent to the horizontal axis in the topology of the Hausdorff uniformity. This is due to the fact that this topology has not a good behavior with respect to unbounded sets.

A coarser topology is the so-called *Attouch-Wets topology* (see [2] for a survey). Traditionally (see [1]), this topology is introduced as a topological convergence in a metric space (X,d): a net $\{A_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}_0(X)$ is said to be *Attouch-Wets* convergent to the nonempty set A if for every nonempty bounded subset $B \subseteq X$ and every $\varepsilon > 0$

$$A \cap B \subseteq B_d(A_\lambda, \varepsilon) \text{ and } A_\lambda \cap B \subseteq B_d(A, \varepsilon) \text{ residually,}$$

where $B_d(A,\varepsilon) = \{x \in X : d(A,x) < \varepsilon\}$ is the ε-enlargement of A.

A uniform version of the Attouch-Wets topology was considered in [15, Section 6] by means of totally bounded sets, from where a quasi-uniform version can be naturally defined.

The two above topologies follow a pattern that can be generalized. Notice that if we consider the family $\mathcal{P}_0(X)$, then convergence of a net $\{A_\lambda\}_{\lambda\in\Lambda}$ to A in the topology of the Hausdorff quasi-uniformity is equivalent to ask that $A_\lambda \cap B \subseteq U(A)$ and $A \cap B \subseteq U(A_\lambda)$ residually for all $U \in \mathcal{U}$ and $B \in \mathcal{P}_0(X)$. So in both cases, the convergence is constructed by means of the truncation with a certain family of sets: the nonempty subsets in the case of the topology of the Hausdorff quasi-uniformity and the nonempty bounded subsets in the case of the Attouch-Wets topology.

In this way, it is natural to study other convergences expressed in terms of truncations and enlargements with respect to an arbitrary family $\mathcal{S}$ of nonempty subsets of X. The filters which generate these convergences were perhaps first considered by Di Maio, Meccariello and Naimpally in [14, 15, 16] although the first deep study was made by Lechicki, Levi and Spakowski [12] (see [3] for a survey). We present here an asymmetric version of the so-called bornological convergences [21].

Definition. Let $(X,\mathcal{U})$ be a quasi-uniform and $\mathcal{S}$ a family of nonempty subsets of X. We say that a net of nonempty subsets $\{A_\lambda\}_{\lambda\in\Lambda}$:

(1) $\mathcal{S}_{\mathcal{U}}^+$-converges to A if $A_\lambda \cap S \subseteq U(A)$ residually for each $S \in \mathcal{S}$ and $U \in \mathcal{U}$;
(2) $\mathcal{S}_{\mathcal{U}}^-$-converges to A if $A \cap S \subseteq U^{-1}(A_\lambda)$ residually for each $S \in \mathcal{S}$ and $U \in \mathcal{U}$;
(3) $\mathcal{S}_{\mathcal{U}}$-converges to A if $\mathcal{S}_{\mathcal{U}}^-$-converges to A and $\mathcal{S}_{\mathcal{U}}^+$-converges to A.

In the sequel, we will omit the subscript $\mathcal{U}$ if no confusion arises.

It is very easy to see that no different convergence appears if we replace $\mathcal{S}$ by the family of all subsets of finite unions of members of $\mathcal{S}$. Consequently, we will only consider *ideals*, i. e. families of nonempty subsets which are closed under nonempty subsets and finite unions. When an ideal $\mathcal{S}$ is also a cover then it is called a *bornology*. Since bornologies are more usual in applications, this kind of convergences is known as *bornological convergences*, whether or not the ideal is a bornology.

In general, bornological convergences are not topological as the next example shows.

Example. *Let us consider the real line* $\mathbb{R}$ *with the usual metric* d *and the bornology* $\mathcal{S}$ *of nonempty finite subsets. Define* $B_n = \{1\} \cup \{1/k : k \geq n\}$ *for all* $n \in \mathbb{N}$. *It is very easy to show that* $\{B_n\}_{n\in\mathbb{N}}$ *is* $\mathcal{S}$*-convergent to* $\{1\}$ *because given* $S \in \mathcal{S}$ *then* $\{1\} \cap S \subseteq B_n$ *for all* $n \in \mathbb{N}$ *and since* S *is finite there exist* $n_0 \in \mathbb{N}$ *such that* $B_n \cap S \subseteq \{1\}$ *for all* $n \geq n_0$.

On the other hand, define $A_n = \{0, 1\} \cup \{1/k : k \geq n\}$. *Then the constant sequence whose terms are equal to* A_n *is* $\mathcal{S}$*-convergent to* B_n. *In fact, it is clear that* $B_n \cap S \subseteq A_n$ *for all* $S \in \mathcal{S}$ *and if* $0 \in A_n \cap S$ *then* $0 \in (\{1/k : k \geq n\})^\varepsilon$ *for all* $\varepsilon > 0$ *so* $A_n \cap S \subseteq B_n^\varepsilon$.

Nevertheless, the sequence $\{A_n\}_{n\in\mathbb{N}}$ *is not* $\mathcal{S}$*-convergent to* $\{1\}$ *since* $\{0\} \in \mathcal{S}$ *and* $A_n \cap \{0\} = \{0\} \not\subseteq \{1\}^\varepsilon$ *for all* $0 < \varepsilon < 1$. *This means that* $\mathcal{S}$*-convergence does not satisfy the iterated limit condition* [10].

Nevertheless, some results about when a bornological convergence is topological have been obtained [4, 5, 12].

Many authors have wondered if the Hausdorff metric topology and the Wijsman topology are hit-and-miss topologies. This was settled by Naimpally [19] showing that the aforementioned hypertopologies are generalized hit-and-miss topologies.

Here, we will show that bornological convergences are generalized hit-and-miss pretopologies [21]. From this fact, we revisit the recent results about the topologicity of these convergences [4, 21].

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